The Type-problem on the Average for random walks on graphs

Raffaella Burioni*, Davide Cassi†, Alessandro Vezzani‡
Istituto Nazionale di Fisica della Materia, Dipartimento di Fisica
Università di Parma, Parco Area delle Scienze n.7A, 43100 Parma, Italy
(February 6, 2008)

When averages over all starting points are considered, the Type Problem for the recurrence or transience of a simple random walk on an inhomogeneous network in general differs from the usual "local" Type Problem. This difference leads to a new classification of inhomogeneous discrete structures in terms of *recurrence* and *transience on the average*, describing their large scale topology from a "statistical" point of view. In this paper we analyze this classification and the properties connected to it, showing how the average behavior affects the thermodynamic properties of statistical models on graphs.

I. INTRODUCTION

In statistical mechanics and field theory Euclidean lattices describe the geometrical structure of crystals and of more abstract geometrical objects, such as discretized flat space-time. However, most real systems have irregular geometry: this is the case for glasses, polymers, amorphous materials, biological structures and fractals in condensed matter, as well as discretized curved space-times in field theory. The geometrical model for these inhomogeneous systems is a general discrete network made of sites and links, i.e. a graph. From this point of view usual lattices are a particular kind of graphs. Statistical models defined on graphs (e.g. harmonic oscillations, random walks, spin system) are the natural way to describe the physical properties of inhomogeneous real structures.

The study of the relation between geometry and physics is one of the most complex and interesting problem of statistical mechanics and field theory on graphs. The main link between this two aspects is provided by random walks. The latter are usually introduced to describe the diffusion of a classical particle and they are related to Markov chains, potential theory and algebraic graph theory on one side [1], and to many problems of equilibrium and non equilibrium statistical mechanics, disordered systems and field theory on the other [2].

In particular, the large times asymptotics of random walks provides the most effective method to describe the influence of large scale topology on the physical properties of discrete structures. The definition of the spectral dimension for inhomogeneous networks, generalizing the Euclidean dimension of lattices in field theory and phase transitions, is indeed based on long time behavior of random walks [3–5]. More generally, this asymptotic regime allows to classify every graph either as *locally recursive* or *transient*, according to the probability of ever returning to the starting site: the probability is 1 in the former case and less than 1 in the latter, independently of the site. This classification, first introduced by Polya for regular lattices [6], is known as the Type-Problem.

Local transience and recurrence describe local properties of physical models on graphs. However, in the study of statistical models on graphs we are in general interested in average (extensive) thermodynamic quantities. Indeed, while on lattices, due to translation invariance, local quantities are the same on all sites and therefore they are equal to their average, on inhomogeneous structures they depend in general on the site and the average behavior can not be reduced to the local one. In the last few years it has become clear that bulk properties are affected by the average values of random walks return probabilities over all starting sites: this is the case for spontaneous breaking of continuous symmetries [7], critical exponents of the spherical model [8], harmonic vibrational spectra [9]. Therefore the classification of discrete structure in terms of *recurrence on the average* and *transience on the average* appears to be the most suitable. Unfortunately, while for regular lattices the two classifications are equivalent, on more general networks they can be different and one has to study a Type-Problem on the Average [10].

Recently this problem has acquired particular relevance in the study of spin models on graphs. Indeed it has been shown that spontaneous breaking of continuous symmetries occurs at $T > 0$ if and only if the underlying network is transient on the average [11]. Moreover this analysis has shown that relevant and new topological properties of infinite graphs are associated to the *on the average* classification.

*E-mail: burioni@pr.infn.it
†E-mail: cassi@pr.infn.it
‡E-mail: vezzani@pr.infn.it

In this paper we deal with the Type-Problem on the Average and with the topological and thermodynamic properties arising from it, proving some basic theorems and discussing their relevance for present and future development in statistical physics.

The paper is organized as follows. In the next section we introduce the basic concepts and notations concerning random walks on infinite graphs. In section III we analyze local recurrence and transience properties of random walks, defined by the asymptotic behavior of return probabilities generating function. In section IV we consider thermodynamic averages on infinite graphs and in section V we introduce the topological classification in terms of average properties of random walks, based on transience and recurrence *on the average*. In section VI we analyze the general relations between average generating functions of return probabilities and consider a further classification holding for transient on the average graphs, which completes the topological description of infinite graphs in terms of random walk behavior. In section VII and VIII we show the relevance of this topological classification in the study of thermodynamic properties of statistical models on inhomogeneous structures. A summary and a discussion of our results are presented in section IX.

II. RANDOM WALKS ON INFINITE GRAPHS

Let us begin by recalling the basic definitions and results concerning graph theory and random walks on infinite graphs, which will be used in the following. A more detailed and complete treatment can be found in the mathematical reviews by Woess [1,12].

A graph $\mathcal{G}$ is a countable set V of vertices (or sites) (i) connected pairwise by a set E of unoriented links (or bonds) $(i,j) = (j,i)$. If the set V is finite, $\mathcal{G}$ is called a finite graph and we will denote by N the number of vertices of $\mathcal{G}$. A subgraph $\mathcal{S}$ of $\mathcal{G}$ is a graph whose set of vertices $S \subseteq V$ and whose set of links $E' \subseteq E$.

A path in $\mathcal{G}$ is a sequence of consecutive links $\{(i,k)(k,h)\dots(n,m)(m,j)\}$ and a graph is said to be connected, if for any two points $i,j \in V$ there is always a path joining them. In the following we will consider only connected graphs.

The graph topology can be algebraically represented introducing its adjacency matrix A_{ij} given by:

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases} \tag{1}$$

The Laplacian matrix Δ_{ij} is defined by:

$$\Delta_{ij} = z_i \delta_{ij} - A_{ij} \tag{2}$$

where $z_i = \sum_j A_{ij}$, the number of nearest neighbors of i, is called the coordination number of site i. Here we will consider graphs with $z_{max} = \sup_i z_i < \infty$.

In order to describe disordered structures we introduce a generalization of the adjacency matrix given by the ferromagnetic coupling matrix J_{ij}, with $J_{ij} \neq 0 \iff A_{ij} = 1$ and $\sup_{(i,j)} J_{ij} < \infty$, $\inf_{(i,j)} J_{ij} > 0$. One can then define the generalized Laplacian:

$$L_{ij} = I_i \delta_{ij} - J_{ij} \tag{3}$$

where $I_i = \sum_j J_{ij}$.

Every connected graph $\mathcal{G}$ is endowed with an intrinsic metric generated by the chemical distance $r_{i,j}$ which is defined as the number of links in the shortest path connecting vertices i and j.

Let us now introduce the random walk on a graph $\mathcal{G}$ defining the jumping probability p_{ij} between nearest neighbors sites i and j:

$$p_{ij} = \frac{A_{ij}}{z_i} = (Z^{-1}A)_{ij} \tag{4}$$

where $Z_{ij} = z_i \delta_{ij}$. From (4) the probability of reaching in t steps site j starting from i is given by:

$$P_{ij}(t) = (p^t)_{ij} \quad . \tag{5}$$

Recurrence properties of random walks are studied introducing the probability $F_{ij}(t)$ for a walker starting from i of reaching for the first time in t steps the site $j \neq i$, while $F_{ii}(t)$ is the probability of returning to the starting point i for the first time after t steps and $F_{ii}(0) = 0$. The basic relationship between $P_{ij}(t)$ and $F_{ij}(t)$ is given by:

$$P_{ij}(t) = \sum_{k=0}^{t} F_{ij}(k) P_{jj}(t-k) \tag{6}$$

($t > 0$). From the previous definitions $F_{ij} \equiv \sum_{t=0}^{\infty} F_{ij}(t)$ turns out to be the probability of ever reaching the site j starting from i (or of ever returning to i if $j = i$). Therefore $0 < F_{ij} \leq 1$. The generating functions $\tilde{P}_{ij}(\lambda)$ and $\tilde{F}_{ij}(\lambda)$ are given by:

$$\tilde{P}_{ij}(\lambda) = \sum_{t=0}^{\infty} \lambda^t P_{ij}(t) \quad \tilde{F}_{ij}(\lambda) = \sum_{t=0}^{\infty} \lambda^t F_{ij}(t) \tag{7}$$

where λ is a complex number. From definition (7) and from the property $0 < F_{ij} \leq 1$ by Abel theorem we have that $\tilde{F}_{ij}(\lambda)$ is a uniformly continuous function for $\lambda \in [0, 1]$ and $0 < \tilde{F}_{ij}(\lambda) \leq 1$, while $\tilde{P}_{ij}(\lambda)$ is continuous for $\lambda \in [0, 1[$ but it can diverge for $\lambda \to 1^-$.

Multiply equations (6) by λ^t and then summing over all possible t with the initial condition $P_{ij}(0) = \delta_{ij}$ we get the basic relations between $\tilde{P}_{ij}(\lambda)$ and $\tilde{F}_{ij}(\lambda)$

$$\tilde{P}_{ij}(\lambda) = \tilde{F}_{ij}(\lambda) \tilde{P}_{jj}(\lambda) + \delta_{ij} \tag{8}$$

In the following we will call $\tilde{P}_i(\lambda) \equiv \tilde{P}_{ii}(\lambda)$ and $\tilde{F}_i(\lambda) \equiv \tilde{F}_{ii}(\lambda)$.

Before discussing recurrence and transience properties we briefly recall the definition of the Gaussian model on a graph, whose deep relation with random walks will be exploited in the next section.

The Gaussian model on the graph $\mathcal{G}$ can be defined [5] introducing the field ϕ_i which are the functions of $l^\infty(V) = \{(\phi_i)_{i \in V} : \sup_i |\phi_i| < \infty\}$. It exist a unique Gaussian probability measure $d\mu_g \phi$ on $l^\infty(V)$ with mean zero and covariance $(L + \mu)^{-1}$ [5] (μ_{ij} is the diagonal matrix $\mu_{ij} = \mu \delta_{ij}$, $\mu > 0$); $d\mu_g(\phi)$ characterize the Gaussian model and we will write:

$$\langle F(\phi) \rangle = \int F(\phi) d\mu_g(\phi) \tag{9}$$

and in particular:

$$\langle \phi_i \phi_j \rangle = (L + \mu)^{-1}_{ij} \tag{10}$$

Alternately, the Gaussian model can be introduced using standard approach of statistical mechanics via the Hamiltonian:

$$\mathcal{H} = \sum_{i,j \in \mathcal{G}} L_{ij} \phi_i \phi_j + \sum_{i \in \mathcal{G}} \mu \phi_i^2 \tag{11}$$

together with the Boltzmann weight $\exp(-\mathcal{H})$, also leading to (10).

III. LOCAL RECURRENCE AND LOCAL TRANSIENCE

The long time asymptotic behavior of random walks on infinite graphs are determined by the large scale topology of the graph and the quantities $\tilde{F}_i(1)$ and $\lim_{\lambda \to 1} \tilde{P}_i(\lambda)$ can be used to characterize the geometry of the graph itself. In particular a graph is called *locally recurrent* if

$$\tilde{F}_i(1) = 1 \text{ or equivalently } \lim_{\lambda \to 1} \tilde{P}_i(\lambda) = \infty \quad \forall i \tag{12}$$

On the other hand if:

$$\tilde{F}_i(1) < 1 \text{ or equivalently } \lim_{\lambda \to 1} \tilde{P}_i(\lambda) < \infty \quad \forall i \tag{13}$$

the graph is called *locally transient*. By standard Markov chains properties [1] (12) and (13) are independent from the site i and then they can be consider as properties of the graphs.

Let us prove the independence from i of (12). If $\lim_{\lambda \to 1} \tilde{P}_i(\lambda) = \infty$ then by equation (8) we get $\lim_{\lambda \to 1} \tilde{P}_{ji}(\lambda) = \infty$ for all j ($0 < \tilde{F}_{ji}(1) \leq 1$); now from (4) and (5) we have that $z_i P_{ij}(t) = z_j P_{ji}(t)$ and $z_i \tilde{P}_{ij}(\lambda) = z_j \tilde{P}_{ji}(\lambda)$. Then also

$\lim_{\lambda\to 1}\tilde{P}_{ij}(\lambda)=\infty$ and from (8) we obtain $\lim_{\lambda\to 1}\tilde{P}_j(\lambda)=\infty$, $\forall j\in V$. In an analogous way it can be shown that property (13) is independent from the choice of i.

Local transience and local recurrence satisfy important universality properties [1]. Indeed these properties are not modified if we substitute the jumping probabilities of the random walker (4) with the generalized jumping probability:

$$p_{ij}=\frac{J_{ij}}{I_i} \quad . \tag{14}$$

In [1] the invariance of the local recurrence properties under a wide class of transformations of the graph itself is also proven. Local recurrence and transience are not modified by the addition a finite number of links or the introduction of second neighbor links on the graph. Notice that these basic invariance properties prove that local recurrence and transience are determined only by the large scale topology of the graph.

IV. AVERAGES ON INFINITE GRAPHS

Let us now consider thermodynamic averages on infinite graphs. The generalized Van Hove sphere $S_{o,r}\subset\mathcal{G}$ of center o and radius r is the subgraph of $\mathcal{G}$ containing all $i\in\mathcal{G}$ whose chemical distance from o $r_{i,o}$ is $\leq r$ and all the links of $\mathcal{G}$ joining them. We will call $N_{o,r}$ the number of vertices contained in $S_{o,r}$.

The average in the thermodynamic limit $\bar{\phi}$ of a function ϕ_i defined on each site i of the infinite graph $\mathcal{G}$ is:

$$\overline{\phi}\equiv\lim_{r\to\infty}\frac{\sum\limits_{i\in S_{o,r}}\phi_i}{N_{o,r}} \quad . \tag{15}$$

The measure $|S|$ of a subset S of V is the average value $\overline{\chi(S)}$ of its characteristic function $\chi_i(S)$ defined by $\chi_i(S)=1$ if $i\in S$ and $\chi_i(S)=0$ if $i\notin S$. The measure of a subset of links $E'\subseteq E$ is given by:

$$|E'|\equiv\lim_{r\to\infty}\frac{E'_r}{N_{o,r}} \quad . \tag{16}$$

where E'_r is the number of links of E' contained in the sphere $S_{o,r}$. The normalized trace $\overline{\mathrm{Tr}}B$ of a matrix B_{ij} is:

$$\overline{\mathrm{Tr}}B\equiv\overline{b} \tag{17}$$

where $b_i\equiv B_{ii}$. Let us require that:

$$\lim_{r\to\infty}\frac{|\partial S_{o,r}|}{N_{o,r}} \;=\; 0 \tag{18}$$

where $|\partial S_{o,r}|$ is the number of the vertices of the sphere $S_{o,r}$ connected with the rest of the graph.

Under this hypothesis we now prove that the averages of a bounded from below function ϕ_i are independent from the center o of the spheres sequence, using the fact that $\chi_i(S)$ is bounded and that measures of subsets are always well defined. From the boundedness of the coordination number we get for any couple of vertices o and o':

$$N_{o,r}-(z_{max})^{r_{o,o'}}|\partial S_{o',r}|\leq N_{o',r}\leq N_{o,r}+(z_{max})^{r_{o,o'}}|\partial S_{o,r}| \tag{19}$$

and

$$(z_{max})^{-r_{o,o'}}|\partial S_{o,r}|\leq|\partial S_{o',r}|\leq(z_{max})^{r_{o,o'}}|\partial S_{o,r}| \tag{20}$$

Let us consider a bounded from below function ϕ_i. Given two vertices o and o', we have:

$$\frac{\sum\limits_{i\in S_{o',r-r_{o,o'}}}\phi_i+\sum\limits_{i\in S_{o,r}\Delta S_{o',r-r_{o,o'}}}\phi_i}{N_{o,r}}=\frac{\sum\limits_{i\in S_{o,r}}\phi_i}{N_{o,r}}=\frac{\sum\limits_{i\in S_{o',r+r_{o,o'}}}\phi_i-\sum\limits_{i\in S_{o',r+r_{o,o'}}\Delta S_{o,r}}\phi_i}{N_{o,r}} \tag{21}$$

where $S_{o,r}\subseteq S_{o',r+r_{o,o'}}$, $S_{o',r-r_{o,o'}}\subseteq S_{o,r}$ and $S_{o,r}\Delta S_{o',r-r_{o,o'}}$ is the symmetric difference between $S_{o,r}$ and $S_{o,r-r_{o,o'}}$. $|S_{o,r}\Delta S_{o',r-r_{o,o'}}|$ denotes the number of vertices of $S_{o,r}\Delta S_{o',r-r_{o,o'}}$ and from (19) we get:

$$|S_{o,r}\Delta S_{o',r-r_{o,o'}}| \leq (z_{max})^{r_{o,o'}}|\partial S_{o,r}| \qquad |S_{o,r}\Delta S_{o',r-r_{o,o'}}| \leq (z_{max})^{r_{o,o'}}|\partial S_{o,r-r_{o.o'}}| \tag{22}$$

with an analogous equation holding for $S_{o',r+r_{o,o'}}\Delta S_{o,r}$. Defining $\bar{\phi}=0$ if $\phi_i>0$ for all i and $\bar{\phi}=|min_i\phi_i|$ otherwise, from (21) we have:

$$\frac{\displaystyle\sum_{i\in S_{o',r-r_{o,o'}}}\phi_i - \bar{\phi}|S_{o,r}\Delta S_{o',r-r_{o,o'}}|}{N_{o,r}} \leq \frac{\displaystyle\sum_{i\in S_{o,r}}\phi_i}{N_{o,r}} \leq \frac{\displaystyle\sum_{i\in S_{o',r+r_{o,o'}}}\phi_i + \bar{\phi}|S_{o',r+r_{o,o'}}\Delta S_{o,r}|}{N_{o,r}} \tag{23}$$

with property (18) of $\mathcal{G}$ and inequalities (22) we get:

$$\lim_{r\to\infty}\frac{\displaystyle\sum_{i\in S_{o',r-r_{o,o'}}}\phi_i}{N_{o,r}} \leq \lim_{r\to\infty}\frac{\displaystyle\sum_{i\in S_{o,r}}\phi_i}{N_{o,r}} \leq \lim_{r\to\infty}\frac{\displaystyle\sum_{i\in S_{o',r+r_{o,o'}}}\phi_i}{N_{o,r}} \tag{24}$$

since $N_{o,r}=N_{o',r-r_{o,o'}}+|S_{o,r}\Delta S_{o',r-r_{o,o'}}|=N_{o',r+r_{o,o'}}-|S_{o,r}\Delta S_{o',r-r_{o,o'}}|$ using again (18) and (22) we get:

$$\lim_{r\to\infty}\frac{\displaystyle\sum_{i\in S_{o',r-r_{o,o'}}}\phi_i}{N_{o',r-r_{o,o'}}} \leq \lim_{r\to\infty}\frac{\displaystyle\sum_{i\in S_{o,r}}\phi_i}{N_{o,r}} \leq \lim_{r\to\infty}\frac{\displaystyle\sum_{i\in S_{o',r+r_{o,o'}}}\phi_i}{N_{o',r+r_{o,o'}}} \tag{25}$$

Therefore, if the limit with the spheres centered in o' exists, it gives the same result using as center any vertex o.

V. RECURRENCE AND TRANSIENCE ON THE AVERAGE

The study of thermodynamic properties of statistical models on infinite graphs requires the introduction of averages of local quantities. The latter are related to random walks by the return probabilities on the average $\bar{P}$ and $\bar{F}$, which are defined by:

$$\bar{P}=\lim_{\lambda\to 1}\overline{\tilde{P}(\lambda)} \tag{26}$$

$$\bar{F}=\lim_{\lambda\to 1}\overline{\tilde{F}(\lambda)} \tag{27}$$

A graph $\mathcal{G}$ is called *recurrent on the average* (ROA) if $\bar{F}=1$, while it is *transient on the average* (TOA) when $\bar{F}<1$.

Recurrence and transience on the average are in general independent of the corresponding local properties. The first example of this phenomenon occurring on inhomogeneous structures was found in a class of infinite trees called NTD (Fig. 1) which are locally transient but recurrent on the average [10].

Moreover, while for local probabilities (8) gives:

$$\tilde{P}_i(\lambda)=\tilde{F}_i(\lambda)\tilde{P}_i(\lambda)+\ 1 \tag{28}$$

an analogous relation for (27) and (26) does not hold since averaging (28) over all sites i would involves the average of a product, which due to correlations is in general different from the product of the average. Therefore the double implication $\tilde{F}_i(1)=1 \Leftrightarrow \lim_{\lambda\to 1}\tilde{P}_i(\lambda)=\infty$ is not true. Indeed there are graphs for which $\bar{F}<1$ but $\bar{P}=\infty$ (an example in shown in Fig. 2) and the study of the relation between $\bar{P}$ and $\bar{F}$ is a non trivial problem.

VI. PURE AND MIXED TRANSIENCE ON THE AVERAGE

In this section we study the relation between $\bar{P}$ and $\bar{F}$ and we show that a complete picture of the behavior of random walks on graphs can be given by dividing transient on the average graphs into two further classes, which will be called *pure* and *mixed* transient on the average (TOA).

First, considering a ROA graph, we prove that if $\bar{F}=1$ then $\bar{P}=\infty$. In this case for each $\delta>0$ it exists ϵ such that if $1-\epsilon\leq\lambda<1$, we have: $1-\delta\leq\overline{\tilde{F}(\lambda)}\leq 1$. Let us consider the subset $S\subseteq V$ of the sites i such that $\tilde{F}_i(1-\epsilon)<1-\sqrt{\delta}$ and we call $\bar{S}$ its complement. We obtain:

$$1-\delta\leq\overline{\tilde{F}(1-\epsilon)}=\overline{\chi(S)\tilde{F}(1-\epsilon)}+\overline{\chi(\bar{S})\tilde{F}(1-\epsilon)}\leq(1-\sqrt{\delta})|S|+|\bar{S}|=1-\sqrt{\delta}|S| \tag{29}$$

From (29) we get $|S|\leq\sqrt{\delta}$ and then $|\bar{S}|\geq 1-\sqrt{\delta}$. Exploiting the property that $\overline{\tilde{P}(\lambda)}$ is an increasing function of λ, for each $\lambda\geq 1-\epsilon$ we get:

$$\overline{\tilde{P}(\lambda)}\geq\overline{\tilde{P}(1-\epsilon)}\geq\overline{\chi(\bar{S})(1-\tilde{F}(1-\epsilon))^{-1}}\geq|\bar{S}|\delta^{-1/2}\geq(1-\sqrt{\delta})\delta^{-1/2} \tag{30}$$

In this way we proved that for arbitrary large value of $(1-\sqrt{\delta})\delta^{-1/2}$ $(\delta\to 0)$, it exists ϵ such that for each λ, $1-\epsilon\leq\lambda<1$, we have $\overline{\tilde{P}(\lambda)}\geq(1-\sqrt{\delta})\delta^{-1/2}$, and therefore $\bar{P}=\lim_{\lambda\to 1}\overline{\tilde{P}(\lambda)}=\infty$.

Notice that this proof can be easily generalized to graphs in which there is a positive measure subset S such that: $\lim_{\lambda\to 1}\overline{\chi(S)\tilde{F}(\lambda)}=|S|$. Indeed in an analogous way it can be proven that:

$$\bar{P}\geq\lim_{\lambda\to 1}\overline{\chi(S')\tilde{P}(\lambda)}=\infty\qquad\forall S'\subseteq S,|S'|>0 \tag{31}$$

We will call *mixed* transient on the average a TOA graphs having a positive measure subset S such that:

$$\lim_{\lambda\to 1}\overline{\chi(S)\tilde{F}(\lambda)}=|S|. \tag{32}$$

while a graph will be called *pure* TOA, if:

$$\lim_{\lambda\to 1}\overline{\chi(S)\tilde{F}(\lambda)}<|S|\qquad\forall S\subseteq V,|S|>0 \tag{33}$$

Examples of mixed and pure TOA graphs are shown respectively in Fig. 2 and Fig. 3. From the previous proof for mixed TOA graphs we have $\bar{P}=\infty$; let us now study the behavior of $\bar{P}$ on pure TOA graphs. We define k as

$$k=\sup_{S\subseteq V,|S|>0}\lim_{\lambda\to 1}\overline{\chi(S)\tilde{F}(\lambda)}|S|^{-1} \tag{34}$$

and since the graphs is pure TOA, $k<1$. For each $0<\lambda'<1$ we introduce $S_{\lambda'}\subseteq V$ as the set of the vertices i such that $\tilde{F}_i(\lambda')>k$. Exploiting the property that $\tilde{F}_i(\lambda)$ is an increasing function of λ we have $\overline{\chi(S_{\lambda'})\tilde{F}_i(\lambda)}>k|S_{\lambda'}|$ and then $\lim_{\lambda\to 1}\overline{\chi(S_{\lambda'})\tilde{F}_i(\lambda)}>k|S_{\lambda'}|$. From (34) we obtain that $S_{\lambda'}$ has zero measure, i. e. it must be $|S_{\lambda'}|=0$. Exploiting the definition (7) we have, for all $i\in V$, $\tilde{P}_i(\lambda)\leq(1-\lambda)^{-1}$ and we obtain for $\overline{\tilde{P}(\lambda')}$:

$$\overline{\tilde{P}(\lambda')}=\overline{\chi(\bar{S}_{\lambda'})\tilde{P}(\lambda')}+\overline{\chi(S_{\lambda'})\tilde{P}(\lambda')}\leq\overline{\chi(\bar{S}_{\lambda'})(1-\tilde{F}(\lambda'))^{-1}}+|S_{\lambda'}|(1-\lambda')^{-1}\leq|\bar{S}_{\lambda'}|(1-k)^{-1}\leq(1-k)^{-1} \tag{35}$$

Taking the limit $\lambda'\to 1$, we have that for pure TOA graphs $\bar{P}$ is finite.

This prove can be generalized to graphs in which there is a positive measure subset S such that for all $S'\subseteq S$, $|S'|>0$, $\lim_{\lambda\to 1}\overline{\chi(S')\tilde{F}(\lambda)}\leq|S'|$ obtaining

$$\lim_{\lambda\to 1}\overline{\chi(S')\tilde{P}(\lambda)}<\infty.\qquad\forall S'\subseteq S,|S'|>0 \tag{36}$$

VII. RANDOM WALKS AND INFRARED PROPERTIES OF THE GAUSSIAN MODEL

The generating function $\tilde{P}_i(\lambda)$ is strictly connected with the correlation functions of the Gaussian model (10) by the following equation:

$$\tilde{P}_i(\lambda)=\sum_{t=0}^{\infty}\lambda^t(Z^{-1}A)^t_{ii}=(1-\lambda Z^{-1}A)^{-1}_{ii}=[Z\lambda^{-1}(\Delta+(1-\lambda)\lambda^{-1}Z)^{-1}]_{ii} \tag{37}$$

and taking the limit $\lambda \to 1$ we have:

$$\lim_{\lambda\to 1} \tilde{P}_i(\lambda) = z_i \lim_{\mu\to 0} (\Delta + \mu Z)^{-1}_{ii} \tag{38}$$

In [5] the invariance of the limit: $\lim_{\mu\to 0}(\Delta+\mu)^{-1}_{ii}$ under a local rescaling of the masses is proven. In particular we have that if $\lim_{\mu\to 0}(\Delta+\mu)^{-1}_{ii}$ is finite than $\lim_{\mu\to 0}(\Delta+\mu Z)^{-1}_{ii} < \infty$ while when the first limit diverges the latter also diverges. Therefore on locally recurrent graphs we have $\lim_{\mu\to 0}(\Delta+\mu)^{-1}_{ii} = \infty \ \forall i$ while for locally transient graphs $\lim_{\mu\to 0}(\Delta+\mu)^{-1}_{ii} < \infty \ \forall i$.

Let us now consider the average generating function. From (37) we get:

$$\lim_{\lambda\to 1} \overline{\tilde{P}(\lambda)} = \lim_{\mu\to 0} \overline{\mathrm{Tr}}[Z(\Delta+\mu Z)^{-1}] \tag{39}$$

and since the connectivity of $\mathcal{G}$ is bounded we get:

$$\lim_{\mu\to 0} \overline{\mathrm{Tr}}(\Delta+\mu Z)^{-1} \le \lim_{\lambda\to 1} \overline{\tilde{P}(\lambda)} \le z_{max} \lim_{\mu\to 0} \overline{\mathrm{Tr}}(\Delta+\mu Z)^{-1} \tag{40}$$

Exploiting the universality properties of the Gaussian model [9], we have that $\lim_{\mu\to 0}\overline{\mathrm{Tr}}(\Delta+\mu Z)^{-1}$ is finite if and only if $\lim_{\mu\to 0}\overline{\mathrm{Tr}}(L+\mu)^{-1} < \infty$, where L is a generalized Laplacian given by (3). Finally from inequalities (40) we get that $\lim_{\mu\to 0}\overline{\mathrm{Tr}}(L+\mu)^{-1} = \lim_{\mu\to 0}\overline{\langle\phi_i\phi_i\rangle}$ diverges if $\bar{P} = \infty$ i.e. on ROA and mixed TOA graphs, while it is finite on pure TOA graphs, where $\bar{P} < \infty$.

VIII. SEPARABILITY AND STATISTICAL INDEPENDENCE

In this section we prove and discuss an important property characterizing mixed TOA graphs which allows to simplify the study of statistical models on these very inhomogeneous structures. We will show that in this case the graph $\mathcal{G}$ can be always decomposed in a pure TOA subgraph $\mathcal{S}$ and a ROA subgraph $\bar{\mathcal{S}}$ with independent jumping probabilities by cutting a zero measure set of links $\partial\mathcal{S} \equiv \{(i,j) \in E | i \in \mathcal{S}, j \in \bar{\mathcal{S}}\}$. The separability property implies that the two subgraphs are statistically independent and that their thermodynamic properties can be studied separately. Indeed, the partition functions referring to the two subgraphs factorize [11].

As a first step, from definition (32) the set of vertices V of a mixed TOA graph $\mathcal{G}$ can always be decomposed in two complementary subsets S and $\bar{S}$ such that

$$\frac{\overline{\chi(S')\tilde{F}(1)}}{|S'|} < 1 \tag{41}$$

for all $S' \subseteq S$ with $|S'| > 0$ and

$$\frac{\overline{\chi(S'')\tilde{F}(1)}}{|S''|} = 1 \tag{42}$$

for all $S'' \subseteq \bar{S}$ with $|S''| > 0$.

To this decomposition we can associate the two subgraphs $\mathcal{S}$ and $\bar{\mathcal{S}}$ defined as follows: $\mathcal{S}$ has S as set of vertices and its links are all the links $(i,j) \in \mathcal{G}$ such that $i,j \in S$; in the same way $\bar{\mathcal{S}}$ has $\bar{S}$ as set of vertices and its links are all the links $(i,j) \in \mathcal{G}$ such that $i,j \in \bar{S}$. Let us now prove that the measure of the boundary $|\partial\mathcal{S}|$ (16) is zero.

We introduce B_S, the border set of $\mathcal{S}$, defined as the set of the vertices $i \in S$ with $(i,j) \in \partial\mathcal{S}$ for some j while we will call $B_{\bar{S}}$ the border set of $\bar{\mathcal{S}}$. Proving that $|\partial\mathcal{S}| = 0$ is equivalent to show that the measure of B_S and $B_{\bar{S}}$ is zero. Indeed we have $|\partial\mathcal{S}|_r \le |B_S|_r \le z_{max}|\partial\mathcal{S}|_r$ and $|\partial\mathcal{S}|_r \le |B_{\bar{S}}|_r \le z_{max}|\partial\mathcal{S}|_r$, where $|B_S|_r$ and $|B_{\bar{S}}|_r$ are the number of sites in B_S and $B_{\bar{S}}$ contained in the sphere $S_{o,r}$.

Let us suppose that $\partial\mathcal{S} \ge 0$ and that $|B_S| \ge 0$, $|B_{\bar{S}}| \ge 0$. From (31) and (36) we have:

$$\lim_{\lambda\to 1} \overline{\chi(B_S)\tilde{P}(\lambda)} \le \infty \tag{43}$$

and

$$\lim_{\lambda\to 1} \overline{\chi(B_{\bar{S}})\tilde{P}(\lambda)} = \infty \tag{44}$$

We will now derive a relation between $\overline{\chi(B_S)\tilde{P}(\lambda)}$ and $\overline{\chi(B_{\bar{S}})\tilde{P}(\lambda)}$ which can not be satisfied if (43) and (44) hold, leading to a contradiction. This implies that $|\partial\mathcal{S}| = 0$.

Let us evaluate $\tilde{P}_i(\lambda)$ in a site $i \in B_S$

$$\tilde{P}_i(\lambda) = \sum_t \lambda^t p_{ii}^t = \sum_t \lambda^t \sum_{jk} p_{ik} p_{kj}^{t-2} p_{ji} \geq \sum_t \lambda^t \sum_{j \in B_{\bar{S}}} p_{ij} p_{jj}^{t-2} p_{ji} \tag{45}$$

where in the inequality we do not consider the terms in which $j \neq k$ and $j \notin B_{\bar{S}}$ Exploiting the fact that $p_{ij} \geq 1/z_{max}$ we get:

$$\tilde{P}_i(\lambda) \geq \frac{\lambda^2}{z_{max}^2} \sum_t \lambda^{t-2} \sum_{j \in B_{\bar{S},i}} p_{jj}^{t-2} = \frac{\lambda^2}{z_{max}^2} \sum_{j \in B_{\bar{S},i}} \tilde{P}_{jj}(\lambda) \tag{46}$$

where $B_{\bar{S},i}$ is the set of the nearest neighbors sites of i which belong to $B_{\bar{S}}$. If we take the average over the sites $i \in B_S$ we have:

$$\overline{\chi(B_S)\tilde{P}(\lambda)} \geq \frac{\lambda^2}{z_{max}^2} \lim_{r\to\infty} \frac{\chi_i(B_S)}{N_{o,r}} \sum_{i \in S_{o,r}} \sum_{j \in B_{\bar{S},i}} \tilde{P}_{jj}(\lambda) \geq \frac{\lambda^2}{z_{max}^2} \lim_{r\to\infty} \frac{\chi_j(B_{\bar{S}})}{N_{o,r}} \sum_{j \in S_{o,r}} \tilde{P}_{jj}(\lambda) = \frac{\lambda^2}{z_{max}^2} \overline{\chi(B_{\bar{S}})\tilde{P}(\lambda)} \tag{47}$$

If we take the limit $\lambda \to 1$ we have:

$$\lim_{\lambda\to 1} \overline{\chi(B_S)\tilde{P}(\lambda)} \geq \frac{1}{z_{max}^2} \lim_{\lambda\to 1} \overline{\chi(B_{\bar{S}})\tilde{P}(\lambda)} \tag{48}$$

Expressions (48), (43) and (44) can not be satisfied at the same time and therefore one must have $|\partial\mathcal{S}| = 0$

Finally we have to prove that $\mathcal{S}$ is a pure TOA graph and $\bar{\mathcal{S}}$ is a ROA graph, i.e. we introduce the restricted jumping probability on S and $\bar{S}$ p_{ij}^S and $p_{ij}^{\bar{S}}$, given by $p_{ij}^S = p_{ij}$ if $i,j \in S$, $p_{ij}^S = 0$ otherwise and an analogous definition for $p_{ij}^{\bar{S}}$. Then we show that S and $\bar{S}$ with the new jumping probabilities p_{ij}^S and $p_{ij}^{\bar{S}}$ are respectively pure TOA and ROA.

More generally for a walker on $\mathcal{S}$ starting from i, we call $P_{ij}^S(t)$ the probability of reaching site j in t steps and $F_{ij}^S(t)$ the probability of reaching j for the first time in t steps. We will prove that:

$$\overline{\tilde{P}^S(\lambda)} = \overline{\sum_{t=0}^{\infty} \lambda^t P_i^S(t)} = \overline{\chi(S)\tilde{P}(\lambda)}|S|^{-1} \qquad \overline{\tilde{F}^S(\lambda)} = \overline{\sum_{t=0}^{\infty} \lambda^t F_i^S(t)} = \overline{\chi(S)\tilde{F}(\lambda)}|S|^{-1} \tag{49}$$

where the average of $\tilde{P}^S(\lambda)$ and of $\tilde{F}^S(\lambda)$ is taken considering $\mathcal{S}$ as the whole graph. Analogous equations hold also for $\bar{\mathcal{S}}$. From (49), (41) and (36) we easily obtain that $\bar{P}^S < \infty$ i.e. $\mathcal{S}$ is pure TOA, while if we call $P_{ij}^{\bar{S}}(t)$ and $F_{ij}^{\bar{S}}(t)$ the probabilities for a random walk on $\bar{\mathcal{S}}$, we get $\bar{F}^{\bar{S}} = 1$, i.e. $\bar{\mathcal{S}}$ is ROA.

To prove equations (49) first we have to show that:

$$\overline{\tilde{P}^S(\lambda)} = \sum_{t=0}^{\infty} \lambda^t \overline{P^S(t)} \qquad \lambda < 1 \tag{50}$$

Equation (50) implies that the thermodynamic average and the sum over the discretized times t commute when $\lambda < 1$. To prove (50) notice that for all $\lambda < 1$ we have:

$$\overline{\tilde{P}^S(\lambda)} = \lim_{r\to\infty} \sum_{i \in S_{o,r}} N_{o,r}^{-1} \left(\sum_{t=0}^{\bar{t}} \lambda^t P_i^S(t) + \sum_{t=\bar{t}}^{\infty} \lambda^{\bar{t}} P_i^S(t) \right) = \sum_{t=0}^{\bar{t}} \lambda^t \overline{P^S(t)} + \lim_{r\to\infty} \sum_{i \in S_{o,r}} N_{o,r}^{-1} \sum_{t=\bar{t}}^{\infty} \lambda^t P_i^S(t) \tag{51}$$

Now $\sum_{i \in S_{o,r}} N_{o,r}^{-1} \sum_{t=\bar{t}}^{\infty} \lambda^{\bar{t}} P_i^S(t) \leq \lambda^{\bar{t}}(1-\lambda)^{-1}$ and letting in (51) $\bar{t} \to \infty$ we get (50). Obviously an analogous equation holds also for $F_i^S(t)$, $P_i(t)$ and $F_i(t)$. Then we can prove (49) showing that:

$$\overline{P^S(t)} = \overline{\chi(S)P(t)}|S|^{-1} \qquad \overline{F^S(t)} = \overline{\chi(S)\tilde{F}(t)}|S|^{-1} \tag{52}$$

We define $d(i, B_S)$ as the distance between i and the cutset B_S: $d(i, B_S) = \inf_{k \in B_S} r_{i,k}$ and will call S_t the subset of S such that: $S_t = \{i \in S | d(i, B_S) \leq t\}$, exploiting the boundedness of the coordination number we get:

$$|S_t| < (z_{max})^t |B_S| = 0 \tag{53}$$

since $|B_S| = 0$. Taking the average of $P_i^S(t)$ we have:

$$\overline{P^S(t)} = \overline{\chi(\bar{S}_t) P^S(t)} + \overline{\chi(S_t) P^S(t)} \tag{54}$$

Now $\overline{\chi(S_t)P^S(t)} \leq |S_t| = 0$, and then $\overline{P^S(t)} = \overline{\chi(\bar{S}_t)P^S(t)}$. Finally exploiting the fact that on $\bar{S}_t$ we have $\overline{P_i^S(t)} = \overline{P_{ii}(t)}$, we obtain (52). Following analogous steps we obtain the equality for $\overline{F^S(t)}$ and for the averages $\overline{P^{\bar{S}}(t)}$ and $\overline{F^{\bar{S}}(t)}$ defined on $\bar{S}$.

IX. DISCUSSION AND CONCLUSIONS

In this paper we have presented a systematic mathematical analysis of the Type Problem for random walks on infinite graphs by considering return probabilities averaged over all sites. After showing that *recurrence and transience on the average* (ROA and TOA) do not in general coincide with the corresponding local properties, we prove that TOA has to be splitted in two complementary subcases, the *pure* and the *mixed* one. Then we show that a mixed TOA graph can always be decomposed in a ROA and a pure TOA subgraphs by cutting a zero measure set of links. This property has deep physical implications, since it allows to decompose a statistical model defined on a mixed TOA graph in two thermodynamically independent models defined respectively on the ROA and pure TOA subgraphs.

In conclusion, we introduced an exhaustive classification of infinite networks in terms on their average recurrence and transience properties, stating the Type Problem on the Average. This classification is the relevant one in the study of thermodynamic properties of statistical models on inhomogeneous structures.

---

[1] W. Woess *Bull. London Math. Soc.* **26**, (1994) 1-60
[2] J.P. Bouchaud and A. Georges, *Phys. Rep.* **195**, (1990) 127
[3] S. Alexander and R. Orbach, *J. Phys.(Paris) Lett.* **43** (1982) L625
[4] E.W. Montrol and G.H. Weiss, *Journ. of Math. Phys.* **6** (1965) 167
[5] K. Hattori, T. Hattori and H. Watanabe, *Prog. Theor. Phys. Suppl.* **92** (1987) 108
[6] G. Polya, *Math. Ann.* **84**, (1921) 149
[7] D. Cassi, *Phys. Rev.Lett.* **76** (1996) 2941
[8] D. Cassi and L. Fabbian *J.Phys. A* **32** (1999) L93
[9] R. Burioni and D. Cassi *Phys. Rev. Lett.* **76** (1996), 1091
[10] R. Burioni, D. Cassi and S. Regina, *Mod. Phys.Lett. B* **10**, (1996) 1059; *Mod. Phys. Lett.* **B 11** (1997), 1095
[11] R. Burioni, D.Cassi and A. Vezzani *Phys. Rev. E* **60** (1999) 1500; *J.Phys. A* **32** (1999) 5539
[12] B. Mohar and W. Woess *Bull. London Math. Soc.* **21**, (1989) 209-234

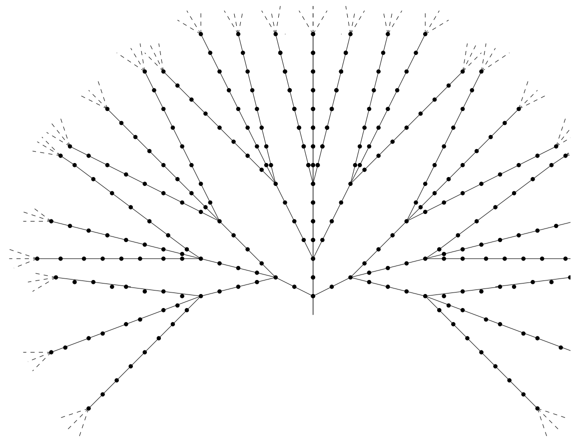

Fig. 1: The NTD tree: the distances between the ramifications increase exponentially. This graph is locally transient and recurrent on the average.

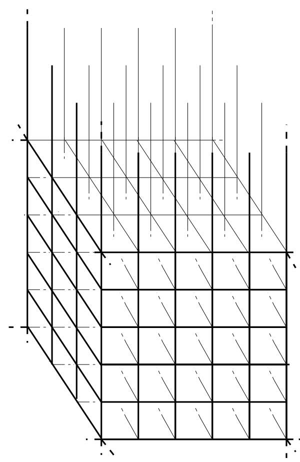

Fig 2: A mixed TOA graph: the cubic lattice is a pure TOA graph while the hairs are ROA.

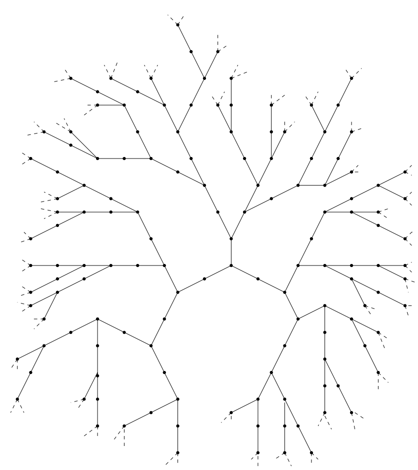

Fig 3: A pure TOA graph, i.e. an inhomogeneous Bethe lattice in which the distance between ramifications can be 1 or 2.